Name KEY Date ______________

Review – Integrals

Show Your Mother ####### Work!

FUNDAMENTAL THEOREM OF CALCULUS
$$\int_a^b f(x)dx = F(b) - F(a)$$
Provided that $f(x)$ is continuous on $[a, b]$ and $F(x)$ is the anti-derivative of $f(x)$.

Calculator Inactive

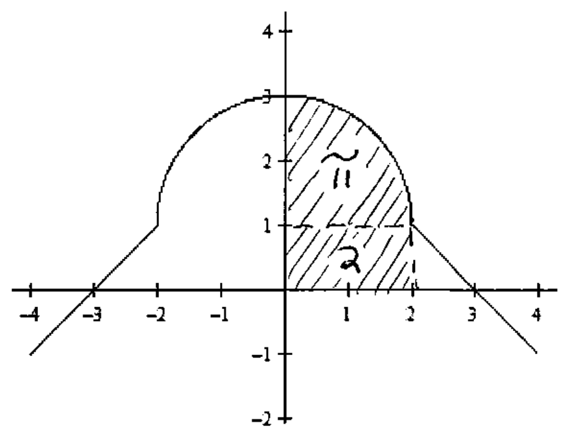
1. If $g(x) = x^2 - 3x + 4$ and $f(x) = g'(x)$, then $\int_1^3 f(x)dx =$

$\int_1^3 f(x)\,dx = g(3) - g(1)$ B/C g IS THE ANTIDERIVATIVE OF f

$4 - 2$

$\boxed{2}$

2. Given to the right is the graph of the derivative of a function, $y = f'(x)$. If $f(0) = 9$, what is the value of $f(2)$?



$\int_0^2 f'(x)\,dx = f(2) - f(0)$

AREA ON INTERVAL (0, 2)

$\pi + 2 = f(2) - 9$

$\boxed{\pi + 11 = f(2)}$

Calculator Active

3. At time $t = 0$ water begins leaking from a tank at the rate of $L(t) = 5e^{-\frac{(t-3)^2}{2}}$ gallons per minute, where t is measured in minutes. How much water has leaked out of the tank after 5 minutes?

$\int_0^5 L(t)\,dt = \boxed{12.231 \text{ GALLONS}}$

SECOND FUNDAMENTAL THEOREM OF CALCULUS

$$\text{If } F(x) = \int_a^{g(x)} f(t)\,dt, \text{ then } F'(x) = f(g(x)) \cdot g'(x)$$

Provided that a is any constant and the lower limit of the interval

Calculator Inactive

4. If f is the function given by $f(x) = \int_4^{2x} \sqrt{t^2 - t}\,dt$, then $f'(2) =$

$f'(x) = \sqrt{(2x)^2 - (2x)} \cdot 2$

$f'(2) = \sqrt{16 - 4} \cdot 2$

$f'(2) = 2\sqrt{12}$

5. The graph of a differentiable function f is shown at right. If $h(x) = \int_0^x f(t)dt$, which of the following is true?

$h(6) = \int_0^6 f(t)\,dt < 0$

$h'(6) = f(6) = 0$

$h''(6) = f'(6) > 0$

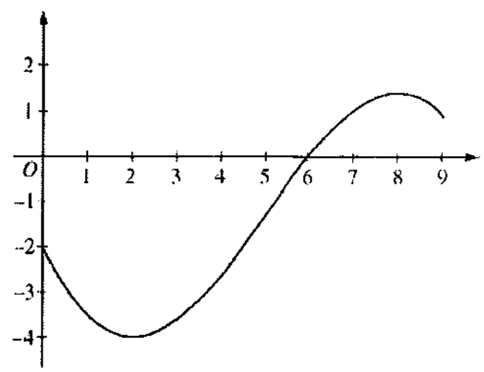
$\boxed{\text{a.}}$ $h(6) < h'(6) < h''(6)$

b. $h(6) < h''(6) < h'(6)$

c. $h'(6) < h(6) < h''(6)$

d. $h''(6) < h(6) < h'(6)$

e. $h''(6) < h'(6) < h(6)$



Graph of f

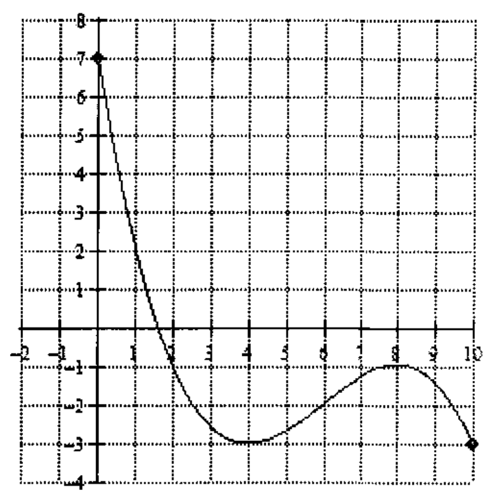
6. The graph of a function f is shown in the figure below and has a horizontal tangent at $x = 4$ and $x = 8$. If $g(x) = x^2 - \int_0^{2x} f(t)\,dt$, what is the value of $g'(3)$?

$g'(x) = 2x - f(2x) \cdot 2$

$g'(3) = 2(3) - 2f(6)$

$g'(3) = 6 - 2(-2)$

$\boxed{g'(3) = 10}$



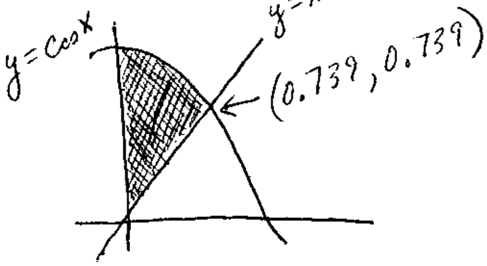
AREA BETWEEN TWO CURVES

$$A = \int_a^b f(x) - g(x)\,dx$$

Where $f(x)$ is the function on top and $g(x)$ is the function on bottom.
*Beware of functions that cross inside the interval (These require two integrals)

Calculator Active

7. Find the area of the region in the first quadrant enclosed by the graphs of $y = \cos x$, $y = x$ and the y–axis.



$$\int_0^{0.739} \cos x - x \; dx = \boxed{0.400}$$

TOP FUNCTION

BOTTOM FUNCTION

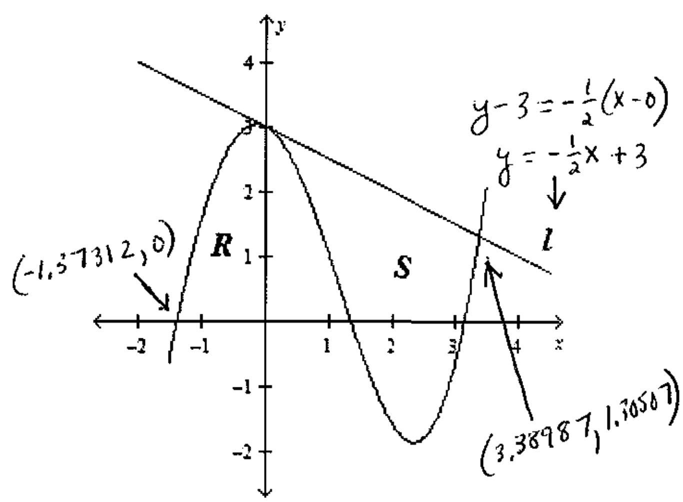
Pictured to the right is the graph of $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$ and a line, l, which is tangent to $f(x)$ at the point (0, 3).

$g(x)$

$f'(0) = -\frac{1}{2}$

8. Find the area of Region R.

$$\int_{-1.37312}^{0} f(x)\; dx = \boxed{2.903}$$



9. Find the area of Region S.

$$\int_0^{3.38987} g(x) - f(x)\; dx = \boxed{6.982}$$

VOLUME OF SOLIDS BY ROTATION

$$V = \pi \int_a^b (Outer\ Function - Axis)^2 - (Inner\ Function - Axis)^2\, dx$$

- When rotating around a vertical line, functions and intervals must be converted to terms of y.

Calculator Inactive

10. A solid is generated when the region in the first quadrant enclosed by the graph of $y = (x^2 + 1)^3$, the line $x = 1$, the x – axis, and the y – axis is revolved about the x – axis. Its volume is found by evaluating which of the following integrals?

$\pi \int_0^1 [(x^2+1)^3 - 0]^2\, dy$

A. $\pi\int_1^8 (x^2+1)^3\, dx$ B. $\pi\int_1^8 (x^2+1)^6\, dx$ C. $\pi\int_0^1 (x^2+1)^3\, dx$

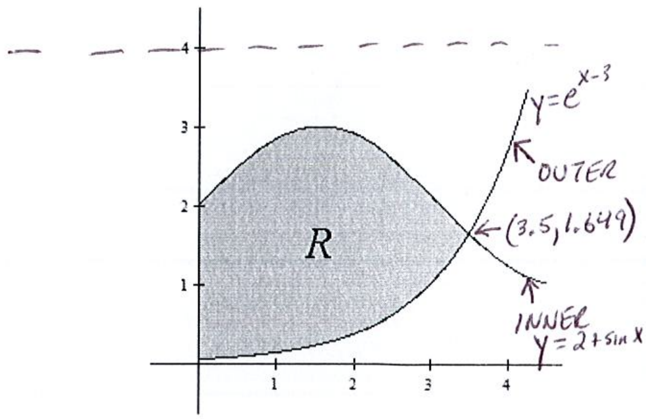
$\boxed{\text{D.}}$ $\pi\int_0^1 (x^2+1)^6\, dx$ E. $2\pi\int_0^1 (x^2+1)^6\, dx$

Calculator Active

11. Let R be the region in the first quadrant bounded by the graphs of $\underbrace{y = 2+\sin x}_{y_1}$, $\underbrace{y = e^{x-3}}_{y_2}$, and the y – axis as shown in the figure above. Find the volume of the solid generated when R is rotated around the line $y = 4$.

$\pi \int_0^{3.5} (y_2 - 4)^2 - (y_1 - 4)^2\, dx$

$\boxed{115.380}$



12. Region R is the region in the first quadrant bounded by the graphs of $f(x) = \sqrt{x}$, $g(x) = 6 - x$ and the x – axis. Find the volume of the solid generated when R is rotated around the line $x = -2$.

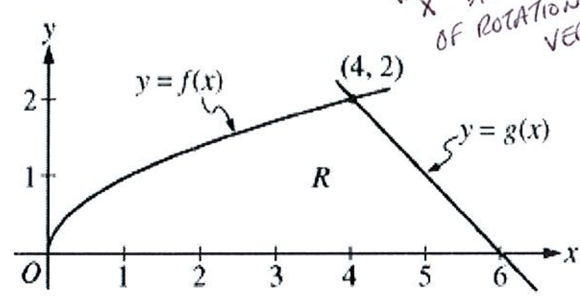
X = -2

* SOLVE THE FUNCTIONS FOR "X" B/C THE AXIS OF ROTATION IS VERTICAL

$f(x) = \sqrt{x}$ $g(x) = 6 - x$

$y = \sqrt{x}$ $y = 6 - x$

$y^2 = x$ $6 - y = x$



$\pi\int_0^2 (6 - y - (-2))^2 - (y^2 - (-2))^2\, dy$

$\boxed{231.221}$

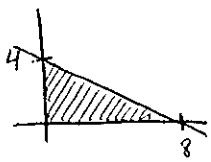
VOLUME OF SOLIDS FORMED BY KNOWN CROSS-SECTIONS

Square	Isosceles Right Triangle	Equilateral Triangle	Semicircle
$V = \int_a^b [f(x) - g(x)]^2\,dx$	$V = \frac{1}{2}\int_a^b [f(x) - g(x)]^2\,dx$	$V = \frac{\sqrt{3}}{4}\int_a^b [f(x) - g(x)]^2\,dx$	$V = \frac{\pi}{8}\int_a^b [f(x) - g(x)]^2\,dx$

Provided the cross-sections are perpendicular to the x – axis. If the cross sections are perpendicular to the y – axis, functions and intervals must be converted to terms of y.

Calculator Inactive

13. The base of a solid is a region in the first quadrant bounded by the x-axis, the y-axis and the graph of the line $x + 2y = 8$. If the cross sections of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid?



$x + 2y = 8$

$y = -\frac{1}{2}x + 4$

$$\frac{\pi}{8}\int_0^8 \left(-\frac{1}{2}x + 4 - 0\right)^2 dx$$

$$\frac{\pi}{8}\int_0^8 \frac{1}{4}x^2 - 4x + 16\,dx$$

$$\frac{\pi}{8}\left(\frac{1}{12}x^3 - 2x^2 + 16x \Big|_0^8\right)$$

$$\frac{\pi}{8}\left(\frac{1}{12}(8)^3 - 2(8)^2 + 16(8) - 0\right)$$

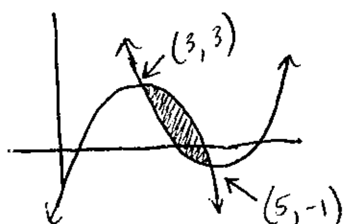
$$\frac{\pi}{8}\left(\frac{8^3}{12} - 2(8)^2 + 16(8)\right)$$

$$\frac{8\pi}{8}\left(\frac{8^2}{12} - \cancel{2(8)} + 16\right)$$

$$\pi\left(\frac{16}{3}\right) = \boxed{\frac{16\pi}{3}}$$

Calculator Active

14. The base of a solid is a region formed by intersecting parabolas $f(x) = -x^2 + 6x - 6$ and $g(x) = x^2 - 10x + 24$. If the cross sections perpendicular to the y – axis are equilateral triangles, what is the volume of the solid?



$\boxed{8.415}$

$y = -x^2 + 6x - 6$

$y = -(x^2 - 6x + 9) - 6 + 9$

$y = -(x-3)^2 + 3$

$y - 3 = -(x-3)^2$

$3 + \sqrt{3-y} = x$

$y = x^2 - 10x + 24$

$y = x^2 - 10x + 25 + 24 - 25$

$y = (x-5)^2 - 1$

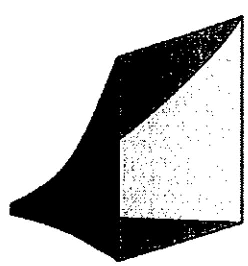
$5 + \sqrt{y+1} = x$

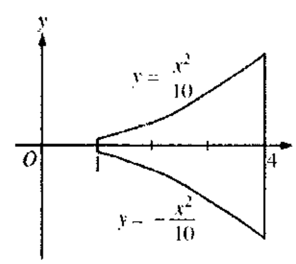
$$\frac{\sqrt{3}}{4}\int_{-1}^{3}\left[\left(3 + \sqrt{3-y}\right) - \left(5 + \sqrt{y+1}\right)\right]^2 dy$$

15. The base of a loud speaker is determined by the two curves $y = \frac{x^2}{10}$ and $y = -\frac{x^2}{10}$ for $1 \le x \le 4$ as shown in the figures to the right. For this loud speaker, the cross sections perpendicular to the x – axis are squares. What is the volume of this speaker, in cubic units?

$$\int_1^4 \left[\frac{x^2}{10} - \left(-\frac{x^2}{10}\right)\right]^2 dx$$

$\boxed{8.184}$



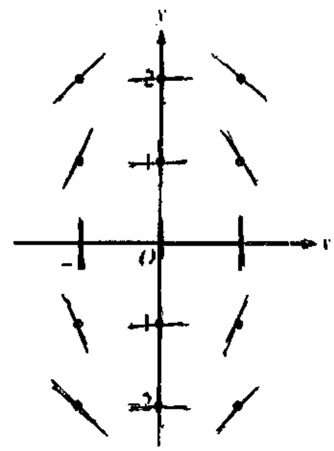


DIFFERENTIAL EQUATIONS & SLOPE FIELDS

$Eg.\ \dfrac{dy}{dx} = \dfrac{x^2}{y}$ ... $(0,-4)$	Differential Equation and given point	Rate of change proportional to amount
$\frac{1}{2}y^2 = \frac{1}{3}x^3 + c$	General Solution	$Eg.\ \dfrac{dy}{dt} = kt$
$f(x) = -\sqrt{\frac{2}{3}x^3 + 16}$	Particular Solution	$y = Ce^{kt}$

16. Consider the differential equation $\dfrac{dy}{dx} = -\dfrac{2x}{y}$. $\left.\dfrac{dy}{dx}\right|_{(-1,2)} = 1$

a. On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.



b. Write an equation of the tangent line to the graph of f at $(1,-1)$ and use it to approximate $f(1.1)$. Explain why the tangent line gives a good approximation of $f(1.1)$.

$\left.\dfrac{dy}{dx}\right|_{(1,-1)} = 2$ $\quad y + 1 = 2(x-1)$

B/C $x = 1.1$ IS CLOSE TO THE P.O.T.

c. Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

$\dfrac{dy}{dx} = -\dfrac{2x}{y}$

$\int y\,dy = \int -2x\,dx$

$\frac{1}{2}y^2 = -x^2 + c$

$\frac{1}{2}(-1)^2 = -(1)^2 + c$

$\frac{1}{2} = -1 + c$

$\frac{3}{2} = c$

$\frac{1}{2}y^2 = -x^2 + \frac{3}{2}$

$y^2 = -2x^2 + 3$

$y = \pm\sqrt{-2x^2 + 3}$

$\boxed{f(x) = -\sqrt{-2x^2 + 3}}$

17. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird in grams at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

a. Is the bird gaining weigh faster when it weighs 40 grams or 70 grams?

WHEN $B = 40g$ ← FASTER

$\dfrac{dB}{dt} = \frac{1}{5}(100 - 40) = 12$ GRAMS/DAY

WHEN $B = 70g$

$\dfrac{dB}{dt} = \frac{1}{5}(100 - 70) = 6$ GRAMS/DAY

b. Find $y = B(t)$, the particular solution to the given differential equation.

$\dfrac{dB}{dt} = \frac{1}{5}(100 - B)$

$\dfrac{1}{100 - B}\,dB = \frac{1}{5}\,dt$

$-\ln|100 - B| = \frac{1}{5}t + c$

$-\ln|100 - 20| = \frac{1}{5}(0) + c$

$-\ln 80 = c$

$-\ln|100 - B| = \frac{1}{5}t - \ln 80$

$\ln|100 - B| = \ln 80 - \frac{1}{5}t$

$100 - B = e^{\ln 80 - \frac{1}{5}t}$

$100 - B = 80e^{-\frac{t}{5}}$

$\boxed{B(t) = 100 - 80e^{-\frac{t}{5}}}$

18. The slope field pictured below represents all general solutions to which of the following differential equations?

~~A.~~ $\frac{dy}{dx} = 2x$

~~B.~~ $\frac{dy}{dx} = -2x$

C. $\frac{dy}{dx} = -y$

D. $\frac{dy}{dx} = y$

E. $\frac{dy}{dx} = x + y$

19. Evaluate the following indefinite integral: $\int \frac{1}{1+x^2}\,dx$

$$\boxed{\tan^{-1}(x) + C}$$

20. Evaluate the following indefinite integral: $\int \frac{3}{\sqrt{1-9x^2}}\,dx$

$$3\int \frac{1}{\sqrt{1-(3x)^2}}\,dx$$

$u = 3x$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$$\frac{1}{3}\cdot 3\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$\sin^{-1} u + C$$

$$\boxed{\sin^{-1}(3x) + C}$$