# The Extreme Value Theorem

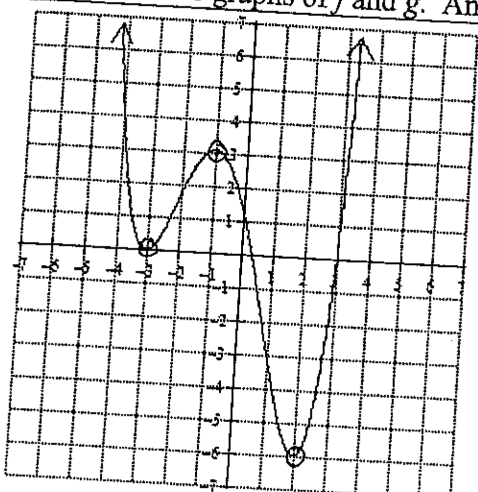
In the previous unit, we investigated heavily how to locate relative extrema of the graph of a function by using the derivative. In pre-calculus, we talked about the difference between relative and absolute extrema. In the space below, distinguish between the two.

**Definitions of Relative and Absolute Extrema of a Function**

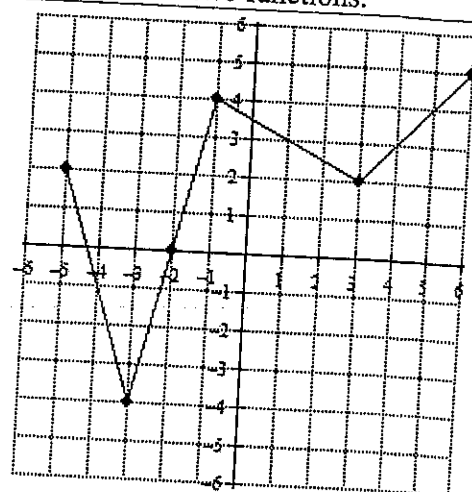
RELATIVE EXTREMA OCCUR WHEN A FUNCTION CHANGES FROM INCREASING TO DECREASING OR VICE VERSA.

ABSOLUTE EXTREMA ARE THE HIGHEST AND LOWEST POINTS ON THE GRAPH OF A FUNCTION.

Pictured below are the graphs of $f$ and $g$. Answer the questions about these two functions.



Graph of $f(x)$



Graph of $g(x)$

Identify the coordinates of the relative extrema of $f$.

REL MIN @ $(-3, 0)$ & $(2, -6)$

REL MAX @ $(-1, 3)$

Identify the coordinates of the relative extrema of $g$.

REL MIN @ $(-3, -4)$ & $(3, 2)$

REL MAX @ $(-1, 4)$

On the domain of $f$, what are the coordinates of the absolute extrema of $f$?

ABS MIN @ $(2, -6)$

NO ABS MAX

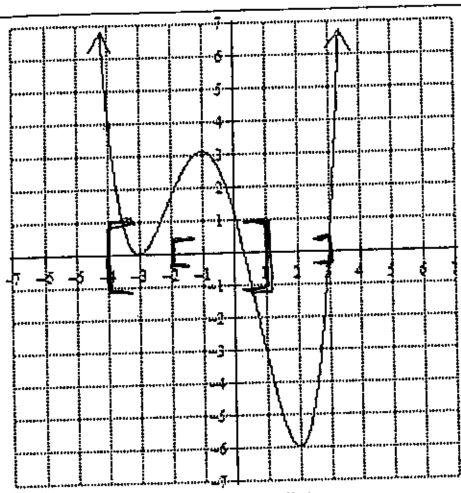
On the domain of $g$, what are the coordinates of the absolute extrema of $g$?

ABS MIN @ $(-3, -4)$

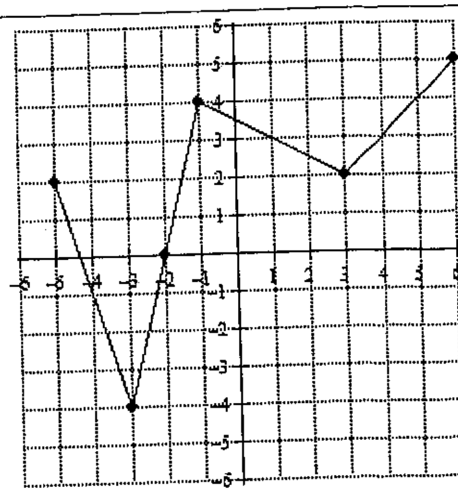
ABS MAX @ $(6, 5)$

On the domain of the given function, did the absolute extrema occur at the function's relative extrema?

SOMETIMES, BUT ABS EXTREMA CAN ALSO OCCUR AT ENDPOINTS



Graph of $f(x)$



Graph of $g(x)$

On the interval $-2 \leq x \leq 3$, what are the absolute extrema of $f$?

ABS MAX $(-1, 3)$

ABS MIN $(2, -6)$

On the interval $-4 \leq x \leq 5$, what are the absolute extrema of $g$?

ABS MAX $(-1, 4)$ & $(5, 4)$

ABS MIN $(-3, -4)$

On the interval $-4 \leq x \leq 1$, what are the absolute extrema of $f$?

ABS MAX $(-4, 4)$

ABS MIN $(1, -3)$

On the interval $-2 \leq x \leq 6$, what are the absolute extrema of $g$?

ABS MAX $(6, 5)$

ABS MIN $(-2, 0)$

When the domain is restricted to a particular closed interval, at what three places that the absolute extrema could exist?

1) THE ENDPOINTS OF THE INTERVAL

2) WHERE $f'(x) = 0$

3) " $f'(x)$ IS UNDEFINED

**The Extreme Value Theorem (E. V. T.):**

IF $f(x)$ IS CONTINUOUS ON THE CLOSED INTERVAL $[a, b]$, THEN $f(x)$ HAS AT LEAST ONE ABSOLUTE MAXIMUM AND AT LEAST ONE ABSOLUTE MINIMUM ON THE CLOSED INTERVAL.

Consider the cubic function $f(x) = -x^3 - 6x^2 - 9x + 2$ to answer the following questions.

a. Determine the intervals where $f$ is increasing and decreasing. Justify your answers.

$$f'(x) = -3x^2 - 12x - 9 = 0$$
$$-3(x^2 + 4x + 3) = 0$$
$$-3(x+3)(x+1) = 0$$

$f'(x)$: $-\infty$ | $-4$: $- \; - \; -$ ⊖ | $-3$ | $-2$: $- \; + \; -$ ⊕ | $-1$ | $0$: $- \; + \; +$ ⊖ | $\infty$

$f(x)$ IS INC. ON $(-3,-1)$ B/C $f'(x) > 0$

$f(x)$ IS DEC. ON $(-\infty,-3) \cup (-1,\infty)$ B/C $f'(x) < 0$

b. Determine the coordinates of the relative extrema of $f$. Justify your answers.

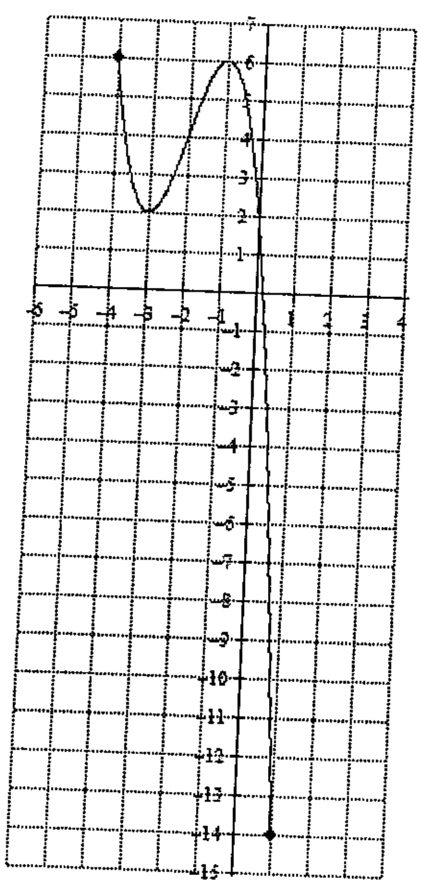
REL MAX @ $x = -1$ B/C $f'(x)$ CHANGES FROM POS TO NEG

$f(-1) = 6$ $\quad (-1, 6)$

REL MIN @ $x = -3$ B/C $f'(x)$ CHANGES FROM $-$ TO $+$

$f(-3) = 2$ $\quad (-3, 2)$

Pictured below is a graph of the function $f$ on the closed interval $-4 \le x \le 1$.



Identify the absolute maximum of $f$ on the closed interval $-4 \le x \le -1$.

$(-4, 6)$ & $(-1, 6)$

Identify the absolute minimum of $f$ on the closed interval $-4 \le x \le -1$.

$(-3, 2)$

Identify the absolute maximum of $f$ on the closed interval $-4 \le x \le 1$.

$(-4, 6)$ & $(-1, 6)$

Identify the absolute minimum of $f$ on the closed interval $-4 \le x \le 1$.

$(1, -14)$

Use the extreme value theorem to locate the absolute extrema of the function $f(x) = -x^3 - 6x^2 - 9x + 2$ on the given closed intervals. Your algebraic results should concur with your graphical conclusions from the previous page.

| Interval: $-4 \le x \le -1$ | Interval: $-4 \le x \le 1$ |
|---|---|
| $f'(x) = -3x^2 - 12x - 9$<br>$f'(x) = -3(x^2 + 4x + 3) = 0$<br>$-3(x+3)(x+1) = 0$<br>$x = -3, -1$<br><br>$f(-4) = 6 \leftarrow$ ABS MAX<br>$f(-3) = 2$<br>$f(-1) = 6$ ABS MIN<br><br>ABS MAX $(-4, 6)$ & $(-1, 6)$<br>ABS MIN $(-3, 2)$ | $f(-4) = 6 \leftarrow$ ABS MAX<br>$f(-3) = 2$<br>$f(-1) = 6$ ABS MIN<br>$f(1) = -14$<br><br>ABS MAX $(-4, 6)$ |

For each of the following functions, state specifically why the E. V. T. is or is not applicable on the given interval.

| Interval: $-5 \le x \le 0$ | |
|---|---|
| $H(x) = \frac{3x+2}{x+3}$ | THE EVT IS NOT APPLICABLE B/C $H(x)$ IS NOT CONTINUOUS @ $X = -3$ WHICH IS ON $[-5, 0]$ |
| $G(x) = 2x\sqrt{x-3}$ | THE EVT IS NOT APPLICABLE B/C $G(x)$ IS ONLY CONTINUOUS WHEN $X \ge 3$ |
| $f(x) = \ln(x+7)$ | THE EVT IS APPLICABLE |

Given the functions below, determine the absolute extreme values of the function on the given interval, provided the extreme value theorem is applicable. If it is not, state specifically why it is not.

1. $f(x) = x^3 - 2x^2 - 3x - 2$ on $[-1, 3]$

CALC. ACTIVE

$f'(x) = 3x^2 - 4x - 3 = 0$

$x = -0.535, 1.869$

$f(-1) = -2$

$f(-0.535) = -1.121 \leftarrow$ ABS MAX

$f(1.869) = -8.065 \leftarrow$ ABS MIN

$f(3) = -2$

ABS MAX $(-0.535, -1.121)$

ABS MIN $(1.86, -8.065)$

2. $g(x) = \sin^2 x + \cos x$ on $\frac{\pi}{2} \le x \le 2\pi$

$g'(x) = 2\sin x \cos x - \sin x$

$g'(x) = \sin x(2\cos x - 1) = 0$

$\sin x = 0 \qquad 2\cos x - 1 = 0$

$x = \cancel{0}, \pi, 2\pi \qquad x = \cancel{\frac{\pi}{3}}, \frac{5\pi}{3}$

$g(\frac{\pi}{2}) = \sin^2(\frac{\pi}{2}) + \cos(\frac{\pi}{2}) = 1$

$g(\pi) = \sin^2(\pi) + \cos(\pi) = -1$ ABS MAX

$g(\frac{5\pi}{3}) = \sin^2(\frac{5\pi}{3}) + \cos(\frac{5\pi}{3}) = \frac{5}{4}$ ABS MIN

$g(2\pi) = \sin^2(2\pi) + \cos(2\pi) = 1$

ABS MAX $(\frac{5\pi}{3}, \frac{5}{4})$ ABS MIN $(\pi, -1)$

3. $f(x) = (x+2)^{\frac{2}{3}}$ on $[-3, 6]$

$f'(x) = \frac{2}{3}(x+2)^{-\frac{1}{3}}$

$f'(x) = \frac{2}{3\sqrt[3]{x+2}}$

UNDEFINED @ $x = -2$

$f(-3) = (-3+2)^{\frac{2}{3}} = 1$

$f(-2) = (-2+2)^{\frac{2}{3}} = 0$ ABS MAX

$f(6) = (6+2)^{\frac{2}{3}} = 4$ ABS MIN

ABS MAX $(6, 4)$

ABS MIN $(-2, 0)$

4. $h(x) = \ln(x^2 - 4)$ on $[-1, 3]$

SINCE $h(x)$ DOES NOT EXIST WHEN $-2 \le x \le 2$, THE EVT IS NOT APPLICABLE